Appendix D: Example Problems

Problem 1: Answer

Integrating over the surface brightness gives the luminosity of a galaxy to be $L \propto I_0 R_0^2$. Because I_0 is constant for all galaxies of this type, $L \propto R_0^2$ for all. The virial theorem implies $M/R_0 \propto v^2$, where M is the mass of a galaxy and v is a typical velocity of stars in a galaxy. Eliminating R_0 gives $L \propto M^2 v^{-4}$. Because the mass-to-light ratio is constant, M/L = constant, so $M \propto L$. Substituting for M in $L \propto M^2 v^{-4}$ gives $L \propto L^2 v^{-4}$, which in turn gives

$$L \propto v^4 \ ,$$

the required result.
This is the same as the Tully-Fisher relation for spiral galaxies, or the Faber-Jackson relation for elliptical galaxies (and observed samples of both types of galaxies do tend to have only a limited range in I_0 and standard $I(R)$ profiles).

Problem 2: Answer

The first galaxy has a strong continuum with absorption lines and some emission lines superimposed. This is characteristic of a spiral galaxy: the first galaxy is a spiral.

The very strong emission lines indicate strong star formation. The star formation rate (which means the mass of interstellar gas forming stars per unit time) is stronger in the second galaxy than in the first. Very strong emission lines on a stellar continuum is a characteristic of an irregular galaxy: the second galaxy might be an irregular galaxy. [However, it could also be a galaxy experiencing a strong burst of star formation because of an interaction with another galaxy, but we shall not consider this here. The simplest explanation is that is an irregular galaxy.]

Problem 3: Answer

When two bodies interact, the interaction is collisional if the interactions between the individual particles in the bodies (molecules for gas, stars for galaxies) affect the motions substantially. The interactions are collisionless if the interactions between the individual particles do not affect the motions.

The interaction is collisional because the interactions between stars are important. This is different to the merger of two galaxies: the star-star encounters are not important in the merger.

Some of the total mechanical energy (potential + kinetic energy) is converted into heat, so the mechanical enery is not conserved. The collapse is dissipative.

Problem 4: Answer

The question gives us the gravitational potential Φ and tells us that the mass distribution has spherical symmetry. To calculate the mass $M(r)$ interior to a radius r we can use the equation

$$M(r) \;=\; \frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

(which applies in cases of spherical symmetry). Differentiating the expression for Φ in the question,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;=\; \frac{G\,M_{tot}\,r}{(r^2+a^2)^{3/2}} \;, \qquad \text{which then gives} \quad M(r) \;=\; \frac{M_{tot}\,r^3}{(r^2+a^2)^{3/2}}$$

(a result that will be given in the lectures).
(We note that $\lim_{r\to\infty} M(r)$ does give M_{tot}, the total mass, as expected.)

[Incidentally, the $M(r) = (r^2/G)\,\mathrm{d}\Phi/\mathrm{d}r$ equation can be derived quite easily. Gauss's Law gives $\int_S \boldsymbol{\nabla}\Phi \cdot \mathrm{d}\mathbf{S} = 4\pi G M_S$ for any closed surface S and for any distribution of mass, where Φ is the potential at a point on the surface and M_S is the total mass enclosed within that surface. Consider the surface S to be a spherical surface of radius r centred on the mass distribution. Therefore $\Phi = \Phi(r)$ is constant over the surface of radius r.
Using a spherical polar coordinate system (r, θ, ϕ) centred on the mass distribution with unit vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$,

$$\boldsymbol{\nabla}\Phi \;\equiv\; \hat{\mathbf{e}}_r\,\frac{\partial\Phi}{\partial r} \;+\; \hat{\mathbf{e}}_\theta\,\frac{1}{r}\,\frac{\partial\Phi}{\partial\theta} \;+\; \hat{\mathbf{e}}_\phi\,\frac{1}{r\sin\theta}\,\frac{\partial\Phi}{\partial\phi} \;=\; \hat{\mathbf{e}}_r\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

in this case because the $\partial\Phi/\partial\theta$ and $\partial\Phi/\partial\phi$ terms are zero on account of the spherical symmetry. So, $\boldsymbol{\nabla}\Phi$ is directed radially outwards and $|\boldsymbol{\nabla}\Phi| = \mathrm{d}\Phi/\mathrm{d}r$.
So $\boldsymbol{\nabla}\Phi$ and $\mathrm{d}\mathbf{S}$ are parallel over the whole surface. Therefore $\boldsymbol{\nabla}\Phi{\cdot}\mathrm{d}\mathbf{S} \;=\; |\boldsymbol{\nabla}\Phi||\mathrm{d}\mathbf{S}|\cos 0$ $= \;|\boldsymbol{\nabla}\Phi|\mathrm{d}S$, which gives in Gauss's Law,

$$|\boldsymbol{\nabla}\Phi| \int_S \mathrm{d}S \;=\; 4\pi\, G\, M(r) \;,$$

using the fact that $\boldsymbol{\nabla}\Phi$ is constant over the surface. Substituting for $|\boldsymbol{\nabla}\Phi| = \mathrm{d}\Phi/\mathrm{d}r$ we get,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,(4\pi r^2) \;=\; 4\pi\,G\,M(r) \qquad \therefore \quad M(r) \;=\; \frac{r^2}{G}\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;,$$

the basic equation used here. But this derivation was not ask for in the question.]

To determine the density ρ, we can consider a thin spherical shell of radius r and thickness $\mathrm{d}r$ centred on the mass distribution. The volume of the shell is $4\pi r^2 \mathrm{d}r$ and its mass is $4\pi r^2 \rho(r)\mathrm{d}r$ where $\rho(r)$ is the density at a radius r. So,

$$\rho(r) \;=\; \frac{1}{4\pi\,r^2}\,\frac{\mathrm{d}M}{\mathrm{d}r} \;,$$

the equation of continuity of mass.
Differentiating the expression for $M(r)$ derived above using the product rule,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{3\,M_{tot}\,r^2}{(r^2+a^2)^{3/2}} - \frac{3\,M_{tot}\,r^4}{(r^2+a^2)^{5/2}} = \frac{3\,M_{tot}\,a^2\,r^2}{(r^2+a^2)^{5/2}} \ .$$

$$\therefore \quad \rho(r) = \frac{3\,M_{tot}}{4\pi}\,\frac{a^2}{(r^2+a^2)^{5/2}} \ ,$$

the result we had to prove.
As an alternative method, we could use Poisson's equation $\nabla^2\Phi = 4\pi G\rho$, which in this case of spherical symmetry gives

$$\rho(r) = \frac{1}{4\pi G}\,\nabla^2\Phi = \frac{1}{4\pi G}\,\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \ .$$

Substituting for the expression for $\mathrm{d}\Phi/\mathrm{d}r$ from above and differentiating would give the required result.

Problem 5: Answer

To find $M(r)$, consider a thin spherical shell of radius r and thickness $\mathrm{d}r$ concentric with the galaxy. The mass in the shell will be

$$\mathrm{d}M = 4\pi\,r^2\,\mathrm{d}r\;\rho(r)$$

(this is the equation of continuity of mass). Integrating from the centre of the galaxy (radius $= 0$) to a radial distance r,

$$\int_0^{M(r)} \mathrm{d}M' = \int_0^r 4\pi\,r'^2\,\mathrm{d}r'\;\rho(r') = \int_0^r 4\pi\,r'^2\,\mathrm{d}r'\;\frac{q\,a}{4\pi}\,\frac{r'^q}{r'^3\,(r'+a)^{q+1}}\;M_{tot} \ .$$

$$\therefore \ M(r) = q\,a\,M_{tot}\int_0^r \frac{r'^{q-1}}{(r'+a)^{q+1}}\,\mathrm{d}r' = M_{tot}\left[\frac{r'^q}{(r'+a)^q}\right]_{r'=0}^{r}$$

on using the standard integral provided. This gives,

$$M(r) = M_{tot}\left(\frac{r^q}{(r+a)^q} - \frac{0^q}{(0+a)^q}\right) = M_{tot}\,\frac{r^q}{(r+a)^q} \ ,$$

the required result (for all $q \neq 0$).

We need to calculate the potential $\Phi(r)$ for $q = 1$ (the Jaffe model). The simplest way to do this is to use

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM(r)}{r^2} \ .$$

From the answer to the first part of the question, putting $q = 1$, the mass $M(r)$ interior to a radius r is

$$M(r) = M_{tot}\,\frac{r}{(r+a)} \qquad \text{for } q = 1.$$

Therefore,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{G}{r^2}\,M_{tot}\,\frac{r}{(r+a)} = \frac{GM_{tot}}{r\,(r+a)} \ .$$

Integrating from a radius r to infinity (remembering that $\Phi(\infty) = 0$ from the definition of gravitational potential),

$$\int_{\Phi(r)}^{0} \mathrm{d}\Phi' \;=\; GM_{tot} \int_{r}^{\infty} \frac{1}{r'\,(r'+a)}\,\mathrm{d}r' \;\; .$$

This can be solved using partial fractions:

$$\begin{aligned}
0 \;-\; \Phi(r) \;&=\; G\,M_{tot} \int_{r}^{\infty} \frac{1}{a} \left(\frac{1}{r'} \;-\; \frac{1}{(r'+a)} \right) \mathrm{d}r' \\
-\,\Phi(r) \;&=\; \frac{G\,M_{tot}}{a} \Big[\ln r' \;-\; \ln(r'+a) \Big]_{r'=r}^{\infty} \\
&=\; \frac{G\,M_{tot}}{a} \left[\ln \left(\frac{1}{1+a/r'} \right) \right]_{r}^{\infty} \;=\; -\,\frac{G\,M_{tot}}{a} \ln \left(\frac{r}{r+a} \right) \\
&=\; \frac{G\,M_{tot}}{a} \ln \left(\frac{r+a}{r} \right) \;\; ,
\end{aligned}$$

which gives

$$\Phi(r) \;=\; -\,\frac{G\,M_{tot}}{a} \ln \left(\frac{r+a}{r} \right)$$

for the potential at a radius r in the Jaffe ($q = 1$) model.

Alternatively, we could approach the problem from a more physical perspective and consider the potential energy released when a particle of mass m is brought from infinity to a radius r in the presence of the gravitational force $F = -GM(r)m/r^2$. The potential energy at a distance r from the centre is then $U_p(r) = m\Phi(r)$, from which we could calculate $\Phi(r)$. This would give the same result as the method above.

If $q \to 0$, the density profile gives $\rho = 0$ for $r > 0$. However,

$$\rho(0) \;=\; \frac{a\,M_{tot}}{4\pi} \lim_{q,\,r \to 0} \frac{q\,r^q}{r^3\,(r+a)^{q+1}} \;\; .$$

So $q \to 0$ implies that all the mass M_{tot} is concentrated at the centre: it corresponds to a point mass.

Problem 6: Answer

The equation of hydrostatic equilibrium contains the pressure gradient $\mathrm{d}P/\mathrm{d}r$. We can relate the pressure to density using the ideal gas law $P = \rho k_B T/m_p$ and therefore get $\mathrm{d}P/\mathrm{d}r$ by differentiating,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\rho\,\frac{k_B T}{m_p}\right) = \frac{k_B T}{m_p}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}$$

because the temperature T and mean mass of each particle m_p are constant throughout (m_p is constant because the chemical composition is the same everywhere). Substituting for $\mathrm{d}P/\mathrm{d}r$ into the equation of hydrostatic equilibrium,

$$\frac{1}{\rho}\,\frac{k_B T}{m_p}\,\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\,\frac{GM(r)}{r^2}\ . \qquad \text{(a)}$$

The equation of continuity of mass provides information about $M(r)$ through

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\,\rho(r)\ ,$$

for any spherically-symmetric distribution of mass. This involves the derivative of $M(r)$. To proceed, we can calculate the derivative of $M(r)$ from equation (a) above so that we can substitute it into the equation of continuity of mass to get an equation involving ρ and r as the only variables. From (a) we get,

$$M(r) = -\,\frac{k_B T}{G m_p}\,\frac{r^2}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\ . \qquad \text{(b)}$$

Differentiating this,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = -\,\frac{k_B T}{G m_p}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\,\frac{k_B T}{G m_p}\left(\frac{2r}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{r^2}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \frac{r^2}{\rho}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}\right)$$

(using the product rule for differentiation). Substituting this into the equation of continuity of mass,

$$-\,\frac{k_B T}{G m_p}\left(\frac{2r}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{r^2}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \frac{r^2}{\rho}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}\right) = 4\pi r^2\,\rho(r)\ .$$

On rearranging this gives

$$\frac{r^2}{\rho}\,\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} - \frac{r^2}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \frac{2r}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{4\pi G m_p}{k_B T}\,r^2\rho = 0\ . \qquad \text{(c)}$$

This is the second-order differential equation involving ρ and r as the only variables that the question asks us to find. (Equation **??** in the course notes expresses this in a slightly different form, but the two expressions are equivalent.)

Try $\rho(r) = \sigma^2/2\pi G r^2$ as a solution to equation (c). Differentiating,

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{\sigma^2}{\pi G r^3} \qquad \text{and} \qquad \frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} = \frac{3\sigma^2}{\pi G r^4} \ .$$

Substituting these into equation (c), the left-hand side of the equation becomes

$$\frac{r^2}{\rho}\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} - \frac{r^2}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \frac{2r}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{4\pi G m_p}{k_B T} r^2 \rho(r)$$

$$= r^2\left(\frac{2\pi G r^2}{\sigma^2}\right)\left(\frac{3\sigma^2}{\pi G r^4}\right) - r^2\left(\frac{2\pi G r^2}{\sigma^2}\right)^2\left(-\frac{\sigma^2}{\pi G r^3}\right)^2$$

$$+ 2r\left(\frac{2\pi G r^2}{\sigma^2}\right)\left(-\frac{\sigma^2}{\pi G r^3}\right) + \frac{4\pi G m_p}{k_B T} r^2 \frac{\sigma^2}{2\pi G r^2}$$

$$= 6 - 4 - 4 + \frac{2 m_p \sigma^2}{k_B T} = -2 + 2\left(\frac{m_p \sigma^2}{k_B T}\right)$$

$$= 0 \quad \text{if} \quad \frac{m_p \sigma^2}{k_B T} = 1 \ , \quad \text{i.e.} \quad \text{if} \quad \sigma = \sqrt{\frac{k_B T}{m_p}} \ .$$

So the left-hand side of equation (c) is equal to the right-hand side if $\sigma = \sqrt{k_B T/m_p}$.
So $\rho(r) = \sigma^2/2\pi G r^2$ is a solution to equation (c) if $\sigma = \sqrt{k_B T/m_p}$.
This is the proof that the question asked for.

To find $M(r)$ we can substitute for $\rho(r) = \sigma^2/2\pi G r^2$ and for $\mathrm{d}\rho/\mathrm{d}r$ into equation (b). This gives,

$$M(r) = -\frac{k_B T}{G m_p} r^2 \left(\frac{2\pi G r^2}{\sigma^2}\right)\left(-\frac{\sigma^2}{\pi G r^3}\right) = \frac{2 k_B T}{G m_p} r \ .$$

Substituting for $k_B T/m_p = \sigma^2$, we get $\quad M(r) = \dfrac{2\sigma^2}{G} r$.

This density profile is known the singular isothermal sphere, discussed in Chapter 2. It was derived here for a gas cloud and applies to gas having a constant temperature T (an isothermal cloud).

[The same density profile, $\rho(r) = \sigma^2/2\pi G r^2$, is often used to model the density of stars in space inside galaxies. When representing stars, σ is the velocity dispersion of the stars about their mean value. However, some caution should be taken when using this potential: the density becomes physically unrealistic at the centre (infinite), while the mass interior to the radius r becomes infinite as $r \to \infty$. This profile should not therefore be used to represent galaxies close to their cores or at very large radii.]

Problem 7: Answer

Because we have spherical symmetry, the magnitude of the gravitational acceleration at a distance r from the centre is $g = GM(r)/r^2$ and this will be equal to the gradient in the gravitational potential, $\mathrm{d}\Phi/\mathrm{d}r$ (from $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ for this spherically symmetric case). Rearranging,

$$M(r) = \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r} \ .$$

Differentiating the expression for $\Phi(r)$ in the question, we get

$$M(r) = 2\pi\rho_0\, a^2 r^2 \left(\frac{2r}{r^2+a^2} - \frac{2a}{r^2}\tan^{-1}\left(\frac{r}{a}\right) + \frac{2a}{r}\frac{a}{r^2+a^2} \right)$$

using the standard result $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(\dfrac{r}{a}\right) = \dfrac{a}{r^2+a^2}$. Rearranging gives,

$$M(r) = 4\pi\rho_0\, a^2 \left(r - a\tan^{-1}\left(\frac{r}{a}\right)\right) ,$$

the result the question asks us to find.

The circular velocity v_{circ} is given by $\dfrac{v_{circ}^2}{r} = |\mathbf{g}| = \dfrac{GM(r)}{r^2}$. Therefore,

$$v_{circ} = \sqrt{\frac{4\pi\rho_0\, a^2}{r}\left(r - a\tan^{-1}\left(\frac{r}{a}\right)\right)} = \sqrt{4\pi\rho_0\, a^2 \left(1 - \frac{a}{r}\tan^{-1}\left(\frac{r}{a}\right)\right)} ,$$

for any value of r.

When $r \gg a$, $\tan^{-1}(r/a) \simeq \pi/2$ and $a/r \ll 1$. Therefore $1 - \frac{a}{r}\tan^{-1}\left(\frac{r}{a}\right) \simeq 1$.

$$\text{So } v_{circ} \simeq \sqrt{4\pi\rho_0\, a^2} , \quad \text{a constant.}$$

So the circular velocity is constant at large radii.
This is what is observed in spiral galaxies, as least as far from the centres that rotation curves can be measured.

For spherical symmetry, the equation of continuity of mass gives $\mathrm{d}M/\mathrm{d}r = 4\pi r^2\rho(r)$, where $\rho(r)$ is the density. Therefore, the density at any r is $\rho(r) = (\mathrm{d}M/\mathrm{d}r)/4\pi r^2$. Differentiating the expression for $M(r)$,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi\rho_0\, a^2 \left(1 - \frac{a^2}{r^2+a^2}\right) = 4\pi\rho_0\, a^2 \left(\frac{r^2}{r^2+a^2}\right)$$

on rearranging. This gives for the density

$$\rho(r) = \frac{1}{4\pi r^2}\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{1}{4\pi r^2} 4\pi\rho_0\, a^2 \left(\frac{r^2}{r^2+a^2}\right) = \frac{\rho_0\, a^2}{r^2+a^2} .$$

This is the "dark matter" profile.
Putting $r = 0$, $\rho(0) = \rho_0$. So ρ_0 is the central density. a is a softening parameter – it makes the profile smoother in the central parts of the galaxy than would be the case if the case if $\rho(r) = k/r^2$, where k is a constant.

The gravitational potential is negative at all times (so that for a mass m the gravitational potential energy $U = m\Phi$ is always negative). So Φ_0 must be a large negative constant for $\Phi(r)$ to be negative.
As $r \to \infty$, $\ln(r^2 + a^2) \to \infty$ and $(2a/r)\tan^{-1}(r/a) \to 0$. Therefore, $\Phi(r) \to \infty$. This is unrealistic as all real potentials should tend to zero as $r \to \infty$ (so that the potential energy of a particle is zero at very large distance).