Queen Mary
University of London

M.Sci. EXAMINATION BY COURSE UNITS

MAS423 Solar System (Semester B, 2007/8)

May 23, 2008, 2:30 pm - 5:30 pm

This paper has two Sections, Section A and Section B: you should attempt both Sections. *Please read carefully the instructions given at the beginning of each section.*

Electronic calculators may be used. The make and model should be specified on the script. The calculator must not be programmed (other than by the manufacturer) prior to the examination.

Potentially useful constants and relations

- Gravitational constant, $G \approx 7 \times 10^{-11}$ N m^2 kg^{-2}
- Specific heat of rock, $c_p \approx 800$ J kg^{-1} K^{-1}
- Melting temperature of rock, $T_{melt} \approx 1000$ K
- Radius of the Earth, $R_{\oplus} \approx 6,000$ km
- Radius of Ceres, $R_{ceres} \approx 500$ km
- Approximate density of rock, $\rho_{rock} \approx 4000$ kg m^{-3}
- Taylor expansion, $(1+x)^n \approx 1 + nx$, $x \ll 1$
- Gradient in polar coordinates $\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}$
- Velocity in polar coordinates, $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{f}\hat{\theta}$

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

SECTION A

Each questions carries 5 marks (2.5 marks for each sub-part). You should attempt ALL five questions.

A1. Describe in one or two sentences what is meant by each of the following terms:

(a) Pericentre

(b) Inclination

A2. For each of these two populations describe briefly (one or two sentences) the orbital characteristics they exhibit, and the orbital domain of the population.

(a) Planetary rings

(b) Irregular satellites

A3. Describe in one or two sentences what is meant by each of the following terms:

(a) Minimum Mass Nebula

(b) Runaway growth

A4. Describe in one or two sentences what is meant by each of the following terms:

(a) Roche zone

(b) Tidal Love number

A5. Describe in one or two sentences what is meant by each of the following terms:

(a) Jacobi Constant

(b) Zero-velocity curves

[Next section overleaf]

SECTION B

Each question carries 25 marks. There are 4 questions.
You may attempt all questions, but only marks for the best 3 questions will be counted.

B1. A satellite of mass m orbits a spherical planet of mass M. The equation of motion of the satellite with respect to the planet is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$$

where $\mu = G(M+m)$, $\mathbf{r}$ is the position vector of the planet with respect to the planet, and $r = |\mathbf{r}|$ is the magnitude of $\mathbf{r}$.

(a) [7 marks] By taking the scalar product of this equation with the velocity vector, $\dot{\mathbf{r}}$, and integrating, show that

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C$$

where $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, and C is a constant of the motion.

(b) [6 marks] The radial distance of the planet is related to its true anomaly, f, by the equation $r = a(1-e^2)/(1+e\cos f)$ where a is the semi-major axis and e the eccentricity of the orbit. Use the expressions for v^2 and r as a function of f given in above and $C = -\mu/2a$ to show that for small values of e,

$$v \approx \sqrt{\frac{\mu}{a}}(1 + e\cos f + e^2).$$

(c) [5 marks] Consider a comet having a close encounter with a planet deep within the planet's Hill sphere. During this parabolic encounter, the comet's distance of closest approach is q. Name two processes, events or mechanisms that might result in the comet becoming permanently captured into a bound orbit about the planet.

(d) [7 marks] Assume that the comet from (c) experiences an instantaneous deceleration at $r = q$. Derive an expression for the magnitude of the minimum change in velocity (Δv) needed to transfer the comet to an orbit entirely contained within the planet's Hill sphere (r_H). Write this expression for Δv in terms of q, r_H, and μ.

[Next question overleaf]

B2. In the planar, circular restricted three-body problem the equations of motion of the massless test particle in the frame rotating with unit angular velocity are given by

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

where the test particle has rectangular coordinates (x, y), in a frame where the x-axis is directed along the line joining the two masses, and

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

with $\mu_1 = m_1/(m_1 + m_2)$, $\mu_2 = m_2/(m_1 + m_2)$, $\mu_1 + \mu_2 = 1$, and it is assumed that $m_2 < m_1$. The square of the distances from the particle to the masses m_1 and m_2 are given by $r_1^2 = (x + \mu_2)^2 + y^2$, $r_2^2 = (x - \mu_1)^2 + y^2$ respectively. In this system, the unit of distance is taken to be the constant separation of the two masses.

(a) [4 marks] Write a brief physical interpretation of the three terms in the definition of the scalar potential U. State the criteria for equilibrium points in this planar system.

(b) [7 marks] Derive expressions for $\partial U/\partial x$ and $\partial U/\partial y$ using the definitions of U, r_1 and r_2 given above. Hence show that the equations of motion have equilibrium solutions at the points $r_1 = r_2 = 1$. Draw a sketch showing the location of these two equilibrium points in relation to the two masses. State, without proof, whether a test particle placed at either of these points is linearly stable or unstable to small displacements?

(c) [4 marks] From the definitions of r_1 and r_2 , show that

$$\mu_1 r_1^2 + \mu_2 r_2^2 = x^2 + y^2 + \mu_1\mu_2$$

and hence derive an expression for U that is an explicit function of r_1 and r_2 only.

(d) [6 marks] An equilibrium point is located close to m_2 along the line joining the two masses such that $r_1 + r_2 = 1$. Using the new expression for U from part (c), or otherwise, show that at this equilibrium point r_2 must satisfy the equation

$$\frac{\mu_2}{\mu_1} = \frac{3r_2^3(1 - r_2 + r_2^2/3)}{(1 - r_2)^3(1 + r_2 + r_2^2)}$$

(e) [4 marks] Find the approximate distance of the equilibrium point from m_2 assuming $m_2 \ll m_1$ and $r_2 \ll 1$. What is this length called?

[Next question overleaf]

B3. This question pertains to energy release and heating during planetary accumulation.

(a) [5 marks] Show that the total energy released when building a body with mass M, radius R, and bulk density (ρ) by impacting planetesimals is at least:

$$E_o = -\frac{3GM^2}{5R} = -\frac{3GR^5}{5}\left(\frac{4\pi}{3}\rho\right)^2$$

(b) [5 marks] Assuming all this energy goes into heat, write an expression for the temperature rise of the planet. Express your answer in terms of density and radius. What is the minimum size a rocky planet must grow to in order to reach the melting temperature by this process? Compare this with the radii of Earth and Ceres and comment on their plausible melting due to this heat source.

(c) [9 marks] Consider that the body differentiates. It continues to have the same bulk density and radius as before, but now has a mantle of density ρ_m overlying a core of radius R_c and density ρ_c. Show that the gravitational potential energy of the differentiated body is

$$E_d = -\frac{3GR^5}{5}\left(\frac{4\pi}{3}\rho_m\right)^2\left(\frac{R_c}{R}\right)^5\left[\left(\frac{\rho_c}{\rho_m}\right)^2 + \left(\frac{R}{R_c}\right)^5 + \frac{5}{2}\left(\frac{\rho_c}{\rho_m} - 1\right)\left(\frac{R^2}{R_c^2} - 1\right) - 1\right]$$

(d) [6 marks] Assume that the core is twice as dense as the mantle and that it occupies half the radius of the body. Show that the gravitational energy released by differentiation is

$$E_o - E_d = -\left(\frac{2}{9}\right)^2 E_o.$$

Calculate the temperature rise due to the energy released in the differentiation process. Is the release of energy due to differentiation sufficient to melt the Earth in this context?

B4. The tidal potential per unit mass experienced by a satellite of mass m moving in a circular orbit of radius a due to the tidal bulge it raises on a homogeneous planet of radius R and mass M is

$$V = -k_2 G \frac{m}{a} \left(\frac{R}{a}\right)^5 P_2(\cos\theta)$$

where k_2 is the Love number of the planet, G is the universal gravitational constant, θ is the lag angle and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the Legendre polynomial of degree 2.

(a) [6 points] Compute the tangential component of the force (i.e., in the $\hat{\theta}$−direction) due to this potential and hence show that the resulting torque on the satellite is

$$\Gamma = G\frac{m^2}{a}\left(\frac{R}{a}\right)^5 \frac{3}{2} k_2 \sin 2\theta.$$

(b) [5 points] Let E be the sum of the rotational energy of the planet and the total orbital energy of the satellite-planet system. Show that $\dot{E}$, the rate of change of energy is given by

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2} m n^2 a \dot{a}$$

where I is the moment of inertia of the planet, Ω is the rotational frequency of the planet and n is the mean motion of the satellite and you may assume $m \ll M$. Since the total energy of the system must be conserved, where does this lost mechanical energy go?

(c) [6 points] Use the conservation of the total angular momentum of the system and the result from (b) to show that

$$\dot{E} = -\frac{1}{2} m a n \dot{a}(\Omega - n).$$

(d) [5 points] Given that $\dot{E} = -\Gamma(\Omega - n) < 0$, use the results above to show that

$$\dot{a} = \frac{3k_2}{Q}\left(\frac{G}{M}\right)^{1/2} R^5 \frac{m}{a^{11/2}}$$

where $Q = 1/\sin 2\theta$ is the tidal dissipation function of the planet. How can this result be used to provide evidence of significant tidal evolution in a system of satellites orbiting a planet?

(e) [3 points] Suggest a means of placing a lower bound on the value of Q of a planet using the masses and orbital elements of its satellites.

[End of examination paper]