Queen Mary

UNIVERSITY OF LONDON

MSci EXAMINATION

ASTM001 Solar System **Duration 3h**

10 May 2004 10.00 – 13.00

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best THREE questions answered will be counted.

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1. Two objects of mass m_1 and m_2 move under their mutual gravitational attraction. The equation of motion defining the variation of the position vector $\boldsymbol{r}$ of the mass m_2 with respect to the mass m_1 is

$$\ddot{\boldsymbol{r}} + \mathcal{G}(m_1 + m_2)\frac{\boldsymbol{r}}{r^3} = 0$$

where $\mathcal{G}$ is the universal gravitational constant.

(a) Taking the vector product of $\boldsymbol{r}$ with the above equation and using the standard result, $\dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ for motion in a polar coordinate system, show that

$$r^2\dot{\theta} = h$$

where h is a constant. Hence show that the rate of change of the area A swept out by the radius vector is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}h\,.$$

(10 marks)

(b) In a polar coordinate system the acceleration vector is given by

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + \left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\right]\hat{\boldsymbol{\theta}}\,.$$

Use the fact that the value of h defined in part (a) is a constant to show that this equation of motion can be written as the scalar equation

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

where $\mu = \mathcal{G}(m_1 + m_2)$. Use the substitution $u = 1/r$ to derive expressions for $\dot{r}$ and $\ddot{r}$ in terms of u, θ and h and hence show that the equation of motion can be written as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}.$$

Write down the solution to this differential equation and hence derive an expression for r in terms of θ. If the boundary conditions are such that the resulting motion is an ellipse, relate any constants of integration to the orbital elements and sketch a plot of r as a function of θ for $0 \leq \theta \leq 2\pi$. (15 marks)

(c) Use the derived expression for $\mathrm{d}A/\mathrm{d}t$ from part (a) and the known geometrical properties of an ellipse to show that the square of the orbital period of the mass m_2 about m_1 is directly proportional to the cube of the semi-major axis of its orbit for elliptical motion. An asteroid is observed to orbit the Sun with a period of 8 years. What is its approximate semi-major axis in astronomical units?

(8 marks)

2. In the planar circular restricted three-body problem approximate equations of motion of the test particle in the rotating frame can be derived in the case where the two masses m_1 and m_2 are such that $m_1 \gg m_2$. In a coordinate system centred on the mass m_2 and rotating at the same rate as the mean motion of the mass m_2 the equations of motion are

$$\ddot{x} - 2\dot{y} = \left(3 - \frac{k}{\Delta^3}\right) x \qquad \text{and} \qquad \ddot{y} + 2\dot{x} = -\frac{k}{\Delta^3} y$$

where k ($\approx m_2/m_1$) is a constant and $\Delta = \sqrt{x^2 + y^2}$ is the distance of the particle from the mass m_2. These are known as Hill's equations and their solution represents the motion of the test particle in the vicinity of the orbiting mass for the circular restricted problem.

(a) Show that the equations of motion can be written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \qquad \text{and} \qquad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$

where $U = (3/2)x^2 + k/\Delta$ and hence show that the quantity

$$C = 3x^2 + 2\frac{k}{\Delta} - \dot{x}^2 - \dot{y}^2$$

is a constant of the motion. How can the existence of C be used to define regions from which the particle will always be excluded? (10 marks)

(b) Calculate the positions of the two equilibrium points associated with the equations of motion and determine the critical value, C_{crit}, of the constant C at these points. Use your knowledge of the full circular restricted problem to sketch the curves of constant C in the vicinity of the mass m_2 in the cases where (i) $C < C_{\text{crit}}$, (ii) $C = C_{\text{crit}}$ and (iii) $C > C_{\text{crit}}$. In each case indicate any region from which the particle is excluded. (12 marks)

(c) The Tisserand relation,

$$\frac{1}{2a} + \sqrt{a(1 - e^2)} \approx \text{ constant}$$

is an approximate constant of the particle's motion in the circular restricted problem, where a and e denote the particle's semi-major axis and eccentricity respectively. Setting $a = 1 + \delta a$ in appropriate units and taking δa and e to be small quantities, use the Tisserand relation to show that

$$3\,\delta a^2 - 4e^2 \approx \text{ constant}.$$

Give a brief description of two practical applications of the Tisserand relation in the solar system. (11 marks)

3. The torque experienced by a satellite of mass m moving in a circular orbit of radius r (about a homogeneous planet of radius R) due to the tidal bulge it raises on the planet is

$$\Gamma = \mathcal{G}\frac{m^2}{r}\left(\frac{R}{r}\right)^5 \frac{3}{2}k_2 \sin 2\theta.$$

where k_2 (a constant) is the Love number of the planet, $\mathcal{G}$ is the universal gravitational constant and θ is the lag angle.

(a) Let E be the sum of the rotational energy of the planet and the orbital energy of the satellite–planet system. Show that $\dot{E}$, the rate of change of this energy, is given by

$$\dot{E} = I\Omega\dot{\Omega} + \frac{1}{2}mn^2 r\dot{r}$$

where I is the moment of inertia of the planet, Ω is the rotational frequency of the planet and n is the mean motion of the satellite. Since the total energy of the system must be conserved, where does this lost energy go?

(5 marks)

(b) Use the conservation of the total angular momentum (rotational plus orbital) of the system and the result from part (a) to show that

$$\dot{E} = -\frac{1}{2}mrn\dot{r}(\Omega - n).$$

(7 marks)

(c) Given that $\dot{E} = -\Gamma(\Omega - n) < 0$, use the results from parts (a) and (b) to show that $\dot{r} \propto r^{-11/2}$ for a given satellite, and give the explicit form of the constant of proportionality. How can this result be used to provide evidence of significant tidal evolution in a system of satellites orbiting a planet?

(10 marks)

(d) The tidal dissipation function, $Q = 1/\sin 2\theta$ is not well determined for most planets. Suggest a mechanism for placing upper and lower bounds on the value of Q for a given planet from observations of the current masses and orbital elements of its satellites, assuming that there has been significant orbital evolution.

(6 marks)

(e) The satellite systems of Jupiter and Saturn contain a large number of mean motion resonances between pairs of satellites. However, there are no known resonances between the major satellites of Uranus. Suggest a mechanism that could explain this observation.

(5 marks)

4. A massless test particle orbits a central star of mass M close to a first-order mean motion resonance with an external perturbing planet which moves on a circular orbit around the star. The averaged part of the disturbing function experienced by the particle due to the perturber is

$$\mathcal{R} = \frac{\mathcal{G}m'}{a'} f(\alpha) e \cos\varphi$$

where $\varphi = j\lambda' + (1-j)\lambda - \varpi$ (with positive integer j) is the resonant angle, $\mathcal{G}$ is the universal gravitational constant, m', a' and λ' are the mass, semimajor axis and mean longitude of the perturber, respectively, e is the eccentricity of the particle's orbit and λ and ϖ are its mean longitude and longitude of pericentre, respectively, and $\alpha = a/a'$ is the ratio of the semimajor axes of the particle and the perturbing planet.

(a) Ignoring secular terms in the expansion, use Lagrange's equations and Kepler's third law to show that

$$\dot{n} = 3(1-j)\mathcal{C}ne\sin\varphi$$

where n is the particle's mean motion and $\mathcal{C} = (m'/M)n\alpha f(\alpha)$. (5 marks)

(b) Ignoring the time variation of e, ϖ and the mean longitude at epoch, show that $\ddot{\varphi} = (1-j)\dot{n}$. Hence show that $\ddot{\varphi}$ satisfies the pendulum equation,

$$\ddot{\varphi} = 3(1-j)^2\mathcal{C}ne\sin\varphi. \qquad \text{(5 marks)}$$

(c) In the case of first-order resonances $\mathcal{C} < 0$, and the energy associated with the pendulum motion of the resonant argument is given by

$$E = \frac{1}{2}\dot{\varphi}^2 - 6(1-j)^2\mathcal{C}ne\sin^2\frac{1}{2}\varphi.$$

Sketch the curves of constant energy in the φ-$\dot{\varphi}$ plane and show that the energy associated with motion on the separatrix (and hence maximum libration) is $E_{\max} = -6(1-j)^2\mathcal{C}ne$. Setting $E = E_{\max}$, derive an expression for $\dot{\varphi}$, and hence show that the maximum variation in mean motion for the test particle in the resonance is

$$\delta n_{\max} = \pm\left(12|\mathcal{C}|ne\right)^{1/2}.$$

What is the corresponding maximum variation in semi-major axis, $\delta a_{\max}$? What is the principal source of error in this estimate when the eccentricities are small and why does it arise? (19 marks)

(d) Given that chaotic motion is associated with the overlap of adjacent resonances, provide a qualitative explanation for the lack of asteroids in the outer par of the asteroid belt. (4 marks)

End of Examination C.D. Murray